数理解析研究所講究録
1010 巻 1997 年 1-6

Modal Intuitionistic Logics and Predicate Superintuitionistic Logics: Correspondence Theory

Guram Bezhanishvili

JAIST

Communicated with H. Ono and N.-Y. Suzuki

Abstract

In this note we deal with intuitionistic modal logics over $\mathcal{MIPC}$ and predicate superintuitionistic logics. We study the correspondence between the lattice of all (normal) extensions of $\mathcal{MIPC}$ and the lattice of all predicate superintuitionistic logics.

Let L_{Prop} denote a propositional language which contains two modal operators $\Box$ and $\Diamond$, and L_{Pred} – a first-order language. Formulas of L_{Prop} and L_{Pred} are built in a usual way. Let us denote the sets of all formulas of L_{Prop} and L_{Pred} by $\mathrm{FORM}(\mathrm{L}_{Prop})$ and $\mathrm{FORM}(\mathrm{L}_{Pred})$ respectively. $\mathcal{MIPC}$ (which was first introduced by A. Prior and R. Bull) is the least subset of $\mathrm{FORM}(\mathrm{L}_{Prop})$ which contains the propositional intuitionistic logic $\mathcal{IN}$, the formulas

$$\begin{array}{ll}
(1)\ \Box p \to p & p \to \Diamond p \\
(2)\ \Box(p \to q) \to (\Box p \to \Box q) & \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q) \\
(3)\ \Diamond p \to \Box\Diamond p & \Diamond\Box p \to \Box p \\
(4)\ \Box(p \to q) \to (\Diamond p \to \Diamond q) &
\end{array}$$

and is closed under substitution, modus ponens and necessitation. A subset of $\mathrm{FORM}(\mathrm{L}_{Prop})$ which contains $\mathcal{MIPC}$ and is closed with respect to those rules of inference is called an *intuitionistic modal logic over* $\mathcal{MIPC}$. Let us denote the lattice of all intuitionistic modal logics over $\mathcal{MIPC}$ by $\Lambda(\mathcal{MIPC})$. For any $\mathcal{L} \in \Lambda(\mathcal{MIPC})$ and $\Gamma \subseteq \mathrm{FORM}(\mathrm{L}_{Prop})$ let $\mathcal{L} \oplus \Gamma$ denote the least logic in $\Lambda(\mathcal{MIPC})$ which contains both $\mathcal{L}$ and Γ.

We will denote by $\mathcal{Q}-\mathcal{IN}$ the standard predicate intuitionistic logic. A subset of $\mathrm{FORM}(\mathrm{L}_{Pred})$ which contains $\mathcal{Q}-\mathcal{IN}$ and is closed with respect to the first-order rules of inference will be called a *predicate superintuitionistic logic.* Let us denote the lattice of all predicate superintuitionistic logics by $\Lambda(\mathcal{Q}-\mathcal{IN})$. For any $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ and

$\Gamma \subseteq \mathrm{FORM}(\mathrm{L}_{Pred})$ let $\mathcal{S} \oplus \Gamma$ denote the least logic in $\Lambda(\mathcal{Q}-\mathcal{IN})$ which contains both $\mathcal{S}$ and Γ.

Following [6], let us fix an individual variable x of L_{Pred}, associate with each propositional variable p of L_{Prop} a unique monadic predicate variable $p^*(\cdot)$ of $\mathrm{FORM}(\mathrm{L}_{Pred})$, and define a translation $\Psi : \mathrm{FORM}(\mathrm{L}_{Prop}) \longrightarrow \mathrm{FORM}(\mathrm{L}_{Pred})$ by putting

$\Psi(p) = p^*(x)$ if p is a propositional variable,

$\Psi(A \circ B) = \Psi(A) \circ \Psi(B)$ where $\circ = \wedge, \vee, \rightarrow$,

$\Psi(\Box A) = \forall x \Psi(A)$,

$\Psi(\Diamond A) = \exists x \Psi(A)$.

It is a routine to check that $\Psi(\mathcal{MIPC}) \subseteq \mathcal{Q}-\mathcal{IN}$. Therefore, Ψ provides us with two morphisms $\Psi^* : \Lambda(\mathcal{MIPC}) \longrightarrow \Lambda(\mathcal{Q}-\mathcal{IN})$ and $\Phi^* : \Lambda(\mathcal{Q}-\mathcal{IN}) \longrightarrow \Lambda(\mathcal{MIPC})$, where $\Psi^*(\mathcal{L}) = \mathcal{Q}-\mathcal{IN} \oplus \Psi(\mathcal{L})$ and $\Phi^*(\mathcal{S}) = \mathcal{MIPC} \oplus \{A \in \mathrm{FORM}(\mathrm{L}_{Prop}) : \Psi(A) \in \mathcal{S}\}$ for any $\mathcal{L} \in \Lambda(\mathcal{MIPC})$ and $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$.

Theorem 1 *(See [7]) 1). Ψ^* and Φ^* are order-preserving morphisms;*
2). Ψ^* is left-adjoint to Φ^*;
3). $\Psi^*(\bigvee_{i\in I} \mathcal{L}_i) = \bigvee_{i\in I} \Psi^*(\mathcal{L}_i)$ for any family $\{\mathcal{L}_i\}_{i\in I} \subseteq \Lambda(\mathcal{MIPC})$.
$\Phi^*(\bigcap_{i\in I} \mathcal{S}_i) = \bigcap_{i\in I} \Phi^*(\mathcal{S}_i)$ for any family $\{\mathcal{S}_i\}_{i\in I} \subseteq \Lambda(\mathcal{Q}-\mathcal{IN})$. □

However, we still do not know whether Ψ^* and Φ^* are complete lattice morphisms or not.

Let us denote by $\mathrm{FORM}(\mathrm{L}_{Pred})_x$ the set of all monadic formulas of L_{Pred} which contain no other individual variable but x. Through Ψ we can identify $\mathrm{FORM}(\mathrm{L}_{Prop})$ with $\mathrm{FORM}(\mathrm{L}_{Pred})_x$. Therefore, for each logic $\mathcal{L} \subseteq \mathcal{MIPC}$, the set $\Psi(\mathcal{L})$ is a subset of $\mathrm{FORM}(\mathrm{L}_{Pred})_x$. For $\mathcal{L} \subseteq \mathcal{MIPC}$ it can happen that there exists $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ such that $\Psi(\mathcal{L})$ coincides with the set of monadic theorems of $\mathcal{S}$ which contain no other individual variable but x. If it is the case we call $\mathcal{L}$ *a one-variable fragment of* $\mathcal{S}$. Note that if $\mathcal{L}$ is a one-variable fragment of $\mathcal{S}$, then $\mathcal{L} \vdash A$ iff $\mathcal{S} \vdash \Psi(A)$ for any $A \in \mathrm{FORM}(\mathrm{L}_{Prop})$. Moreover, as follows from Theorem 2 below, two different logics over $\mathcal{MIPC}$ can not be one-variable fragments of the same predicate superintuitionistic logic. However, as follows from Theorem 4, continuum predicate superintuitionistic logics can have the same one-variable fragment.

A logic $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ is said to be *generated by its one-variable fragment* if the set of monadic theorems of $\mathcal{S}$ which contain no other individual variable but x axiomatizes $\mathcal{S}$. As follows from Theorem 2, two different predicate logics can not be generated by the same one-variable fragment.

Theorem 2 *(Compare with [7]) 1). $\mathcal{L} \in \Lambda(\mathcal{MIPC})$ is a one-variable fragment of a predicate superintuitionistic logic iff $\mathcal{L} \in \Phi^*(\Lambda(\mathcal{Q}-\mathcal{IN}))$ iff $\mathcal{L} = \Phi^*\Psi^*(\mathcal{L})$;*
2). $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ is generated by its one-variable fragment iff $\mathcal{S} \in \Psi^*(\Lambda(\mathcal{MIPC}))$ iff $\Psi^*\Phi^*(\mathcal{S}) = \mathcal{S}$. □

Let us denote the set $\Phi^*(\Lambda(\mathcal{Q}-\mathcal{IN}))$ by Λ_M, and the set $\Psi^*(\Lambda(\mathcal{MIPC}))$ – by Λ_P.

Corollary 3 *(Compare with [7]) 1). Λ_M is a $\bigcap$-sublattice of $\Lambda(\mathcal{MIPC})$ and Λ_P is a $\bigvee$-sublattice of $\Lambda(\mathcal{Q}-\mathcal{IN})$;*

2). $\Psi^*|_{\Lambda_M}$ and $\Phi^*|_{\Lambda_P}$ set an isomorphism between Λ_M and Λ_P, where $\Psi^*|_{\Lambda_M}$ and $\Phi^*|_{\Lambda_P}$ denote the restrictions of Ψ^* and Φ^* to Λ_M and Λ_P respectively;

3). The cardinality of both Λ_M and Λ_P is continuum. □

However, it is still an open question whether Λ_M and Λ_P are complete sublattices of $\Lambda(\mathcal{MIPC})$ and $\Lambda(\mathcal{Q}-\mathcal{IN})$ respectively.

With each logic $\mathcal{L} \in \Lambda_M$ let us associate a family $\Phi^{-1}(\mathcal{L}) = \{\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN}) : \Phi^*(\mathcal{S}) = \mathcal{L}\}$.

Theorem 4 *1). There exist $\mathcal{L} \in \Lambda(\mathcal{MIPC})$ such that the cardinality of $\Phi^{-1}(\mathcal{L})$ is continuum;*

2). $\Phi^{-1}(\mathcal{L})$ has the least element and it is $\Psi^*(\mathcal{L})$. □

However, it is still an open question whether $\Phi^{-1}(\mathcal{L})$ has a greatest element.

Let $\mathcal{S} \in \Lambda_P$ and $\mathcal{S} = \mathcal{Q}-\mathcal{IN} \oplus \Gamma$. Since $\mathcal{S} \in \Lambda_P$, Γ can be regarded as a subset of $\Psi(\mathrm{FORM}(\mathrm{L}_{Prop})) = \mathrm{FORM}(\mathrm{L}_{Pred})_x$. With $\mathcal{S}$ we shall associate a family $\Psi^{-1}(\mathcal{S}) = \{\mathcal{L} \in \Lambda(\mathcal{MIPC}) : \Psi^*(\mathcal{L}) = \mathcal{S}\}$.

Theorem 5 *1). There exist $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ such that the cardinality of $\Psi^{-1}(\mathcal{S})$ is continuum;*

2). $\Psi^{-1}(\mathcal{S})$ has the least element $\Phi^*(\mathcal{S})$ and the greatest element $\mathcal{MIPC} \oplus \{A \in \mathrm{FORM}(\mathrm{L}_{Prop}) : \Psi(A) \in \Gamma\}$. □

As we mentioned before, a logic $\mathcal{L} \in \Lambda(\mathcal{MIPC})$ is a one-variable fragment of a logic $\mathcal{S} \in \Lambda(\mathcal{Q}-\mathcal{IN})$ iff $\mathcal{L} = \Phi^*(\mathcal{S})$. However, we still do not know how to characterize one-variable fragments of predicate superintitionistic logics in internal terms of $\Lambda(\mathcal{MIPC})$[1]. Therefore, we need more knowledge about concrete logics over $\mathcal{MIPC}$ which are one-variable fragments of predicate superintitionistic logics. For this reason we need to develop semantics for both modal intuitionistic logics over $\mathcal{MIPC}$ and predicate superintitionistic logics.

Kripke-type semantics for logics over $\mathcal{MIPC}$ was developed by H. Ono (see [5]). *Ono frames* are triples $\langle W, R, Q\rangle$, where R is a partial order on $W \neq \emptyset$, Q a quasi-order on W such that $R \subseteq Q$ and $\forall w, v \in W\ (wQv \Rightarrow \exists u \in W\ wRu\ \&\ uE_Qv)$, where uE_Qv iff uQv and vQu. An Ono frame $\langle W, R, Q\rangle$ is said to be an *Ono quasi-sheaf* if it satisfies the following condition: $\forall w, v \in W\ (wE_Qv\ \&\ wRv \Longrightarrow w = v)$. An Ono quasi-sheaf is said to be an *Ono sheaf* if $\forall w, v \in W\ (wRv\ \&\ wRu\ \&\ vE_Qu \Longrightarrow v = u)$. The definition of *regular Ono frames* can be found in [7][2].

Let us consider the following list of formulas:

$\mathcal{P}_0:\quad \bot;$

---

[1]Partial results in this direction can be found in [1].

[2]In [7] they were called *regular frames*.

$\mathcal{P}_n$: $q_n \vee (q_n \rightarrow \mathcal{P}_{n-1})$, $n \geq 1$;
$\mathcal{Q}_0$: $\perp$;
$\mathcal{Q}_n$: $\Box(q_n \vee (q_n \rightarrow \mathcal{Q}_{n-1})$, $n \geq 1$;
$\mathcal{D}$: $\Box(\Box p \vee q) \rightarrow \Box p \vee \Box q$;
$\mathcal{K}$: $\Box\neg\neg p \rightarrow \neg\neg\Box p$;
$\mathcal{LIN}$: $(p \rightarrow q) \vee (q \rightarrow p)$.

The following theorem shows the effectiveness of Kripke-type semantics for logics over $\mathcal{MIPC}$.

Theorem 6 *1). ([5]) $\mathcal{MIPC}$ is complete with respect to all regular Ono frames;*

2). ([5]) $\mathcal{MIPC} \oplus \mathcal{D}$ is complete with respect to its Ono frames;

3). ([7]) $\mathcal{MIPC} \oplus \mathcal{K}$ is complete with respect to its regular Ono frames;

4). ([1] and [2]) $\mathcal{MIPC} \oplus \mathcal{LIN}$ is complete with respect to its Ono frames;

5). ([1] and [2]) For each logic $\mathcal{L} \supseteq \mathcal{MIPC}$ if there exists $n \in \omega$ such that $\mathcal{MIPC} \oplus \mathcal{P}_n \subseteq \mathcal{L}$, then $\mathcal{L}$ is complete with respect to its Ono frames. □

An extended Kripke-type semantics for predicate superintuitionistic logics was introduced by V. Shehtman and D. Skvortsov (see [8]). A triple $\langle D, \pi, W\rangle$ is said to be a *Kripke bundle* if $D \neq \emptyset$ and $W \neq \emptyset$ are partially ordered sets and $\pi : D \longrightarrow W$ is a surjective p-morphism. A Kripke-bundle is called a *Kripke quasi-sheaf* if $\forall x, y \in D$ $(\pi(x) = \pi(y)$ & $x \rho y \Longrightarrow x = y)$. A Kripke bundle is called a *Kripke sheaf* if $\forall x \in D$, $\forall w \in W$ $(\pi(x) R w \Longrightarrow \exists! y \in D :$ $x \rho y$ & $\pi(y) = w)$.

There is a close correspondence between Kripke-type semantics for logics over $\mathcal{MIPC}$ and extended Kripke-type semantics for predicate superintuitionistic logics. Indeed, we have the following

Theorem 7 *1). ([2]. A related result was also obtained in [10]) The category of all Ono frames is equivalent to the category of all Kripke bundles;*

2). ([2]) The category of all Ono quasi-sheaves is equivalent to the category of all Kripke quasi-sheaves;

3). ([2]) The category of all Ono sheaves is equivalent to the category of all Kripke sheaves;

4). ([2]. A related result was also obtained in [7]) The category of all regular Ono frames is equivalent to the category of all standard Kripke frames. □

On the base of this theorem we can obtain the following sufficient condition for a logic $\mathcal{L} \supseteq \mathcal{MIPC}$ to be a one-variable fragment of a predicate superintuitionistic logic. For any Ono frame $\mathcal{F} = \langle W, R, Q\rangle$ let $\widetilde{\mathcal{F}}$ denote its corresponding Kripke bundle.

Theorem 8 *1). (Compare with [10] and [7]) If a logic $\mathcal{L} \supseteq \mathcal{MIPC}$ is complete with respect to a class $\{\mathcal{F}_i\}_{i\in I}$ of Ono frames and $\Psi^*(\mathcal{L})$ is sound with respect to $\{\widetilde{\mathcal{F}}_i\}_{i\in I}$, then $\mathcal{L}$ is a one-variable fragment of $\Psi^*(\mathcal{L})$;*

2). If $\mathcal{L} \supseteq \mathcal{MIPC}$ is complete with respect to a class of Ono sheaves, then $\mathcal{L}$ is a one-variable fragment of $\Psi^*(\mathcal{L})$. □

As a result of 2) of Theorem 8 we obtain the following theorem which extends Theorem 4.1 from [7].

Theorem 9 *For any $n \in \omega$ and any $\mathcal{L} \supseteq \mathcal{MIPC} \oplus \mathcal{LIN} \oplus \mathcal{Q}_n$, $\mathcal{L}$ is a one-variable fragment of $\Psi^*(\mathcal{L})$.* □

Note that M. Wajsberg's theorem, which states that $\mathcal{S}5$ is a one-variable fragment of the classical predicate logic $\mathcal{Q}-\mathcal{CL}$, is also a consequence of Theorem 9. The other known examples of one-variable fragments of predicate superintuitionistic logics can be gathered in the following

Theorem 10 *1). (See [3] and [7]) $\mathcal{MIPC}$ is a one-variable fragment of $\mathcal{Q}-\mathcal{IN}$;*
2). (See [5]) $\mathcal{MIPC} \oplus \mathcal{D}$ is a one-variable fragment of $\mathcal{Q}-\mathcal{IN} \oplus \Psi(\mathcal{D})$;
3). (See [7]) $\mathcal{MIPC} \oplus \mathcal{K}$ is a one-variable fragment of $\mathcal{Q}-\mathcal{IN} \oplus \Psi(\mathcal{K})$;
4). (See [9]) For any propositional superintuitionistic logic $\mathcal{J} \supseteq \mathcal{IN}$, $\mathcal{MIPC} \oplus \mathcal{J} \oplus \Diamond p \to \Box p$ is a one-variable fragment of $\mathcal{Q}-\mathcal{IN} \oplus \mathcal{J} \oplus \exists x p^*(x) \to \forall x p^*(x)$. □

The examples of one-variable fragments of predicate superintuitionistic logics mentioned above were mainly obtained on the base of 2) of Theorem 8. For obtaining more examples and for further clarification of correspondence between logics over $\mathcal{MIPC}$ and predicate superintuitionistic logics we need to use a more general Item 1 of Theorem 8 in a more systematic way. For this we need to know which Kripke bundles are closed with respect to substitution. Recall that Shehtman and Skvortsov proved that every Kripke sheaf is closed with respect to substitution. So, we need to extend this result to more larger classes of Kripke bundles. In this direction first should be attacked the following questions:

1). For any propositional superintuitionistic logic $\mathcal{J}$ and a given Kripke bundle $\langle D, \pi, W\rangle$, what is a necessery and sufficient condition for $\langle D, \pi, W\rangle \models \mathcal{Q}-\mathcal{IN} \oplus \mathcal{J}$?

2a). Give a "reasonable" sufficient condition for a Kripke bundle $\langle D, \pi, W\rangle$ which is not a Kripke sheaf to be closed with respect to substitution.

2b). What is a necessery and sufficient condition for a Kripke bundle $\langle D, \pi, W\rangle$ to be closed with respect to substitution?

References

[1] G. Bezhanishvili, *Varieties of monadic Heyting algebras. Part I.*, 1997, Report ♯ IS-RR-97-0001F, JAIST.

[2] G. Bezhanishvili, *Varieties of monadic Heyting algebras. Part II: Duality theory.*, 1997, Report ♯ IS-RR-97-0004F, JAIST.

[3] R.A. Bull, *MIPC as the formalization of an intuitionist concept of modality*, Journal of Symbolic Logic 31(1966), pp. 609-616.

[4] H. Ono, *A study of intermediate predicate logics*, Publications of Research Institute for Mathematical Sciences, Kyoto University 8(1972), pp. 619-649.

[5] H. Ono, *On some intuitionistic modal logics*, Publications of Research Institute for Mathematical Sciences, Kyoto University 13 (1977), pp. 687-722.

[6] H. Ono, *Some problems in intermediate predicate logics*, Reports on Mathematical Logic 21(1987), pp. 55-67.

[7] H. Ono and N.-Y. Suzuki, *Relations between intuitionistic modal logics and intermediate predicate logics*, Reports on Mathematical Logic 22(1988), pp. 65-87.

[8] V.B. Shehtman and D.P. Skvortsov, *Semantics of non-classical first-order predicate logics*, in Mathematical Logic, edited by P. P. Petkov, Plenum Press, New York, 1990, pp. 105-116.

[9] N.-Y. Suzuki, *An algebraic approach to intuitionistic modal logics in connection with intermediate predicate logics*, Studia Logica 48(1989), pp. 141-155.

[10] N.-Y. Suzuki, *Kripke bundles for intermediate predicate logics and Kripke frames for intuitionistic modal logics*, Studia Logica 49(1990), pp. 289-306.